1-1.

DEFINITION 1.1 Differential Equation

An equation containing the derivatives or differentials of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation** (DE).

Type

Order

Linear/non Linear

$$\frac{d[\text{dependent variable}]}{d[\text{independent variable}]}.$$

Type — Ordinary Differential Equation (Math 3?)

contains only ordinary derivatives of one or more dependent var. with respect to a single independent var.

Partial Different Equation

involve the partial derivatives of one or more dependent var partial derivative

" ∂ "

$$u(x,y) = x^2+y^2.$$

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial x^2+y^2}{\partial x} = 2x+0 = 2x$$

$$\frac{\partial u(x,y)}{\partial y} = \frac{\partial x^2+y^2}{\partial y} = 0+2y = 2y.$$

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e.g. $\frac{du}{dx} - \frac{dv}{dx} = x$ ODE u, v : dependent var, depends on x only

$\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} = \frac{\partial^2 v}{\partial x^2}.$ PDE. u, v : dependent var.

u depends on x, y

v depends on x (so far)

Order — order of the eq. is the order of the highest order of the equation.

$$\frac{du}{dx} - \frac{d^2v}{dx^2} = x.$$ ← 2nd order ODE.

first order second order

this is power, not derivative.

$$\frac{du}{dx} - \left(\frac{dv}{dx}\right)^{2} = x$$ ← 1st order ODE.

first order first order

same idea applies for P.D.E as well.

Linearity — Linear.

— Non linear an eq that is not linear.

$$\underbrace{a_n(x)}_{\text{function in term of } x}\frac{d^ny}{dx^n} + a_{n-1}(x)\underbrace{\frac{d^{n-1}y}{dx^{n-1}}}_{(n-1)\text{th derivative}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = \underbrace{g(x)}_{\text{function}}$$

① No power of y and derivative of y greater than 1.

↳ $y' + x = 0$ Linear. $(y'')^2 + y' - x = 0$ not Linear

② no product of $y\frac{dy}{dx}$

each coefficient depends on only the independant variable x.

↳ $a_n(x)$, $a_{n-1}(x)$, ⋯, $a_1(x)$, $a_0(x)$ are coefficients

they only dependent on x

DEFINITION 1.2 Solution of a Differential Equation

Any function f defined on some interval I, which when substituted into a differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

$y = e^{3x} + 10e^{2x}$ is a solution to $\frac{dy}{dx} - 2y = e^{3x}$ ← RHS

↳ $\frac{dy}{dx} = 3e^{3x} + 20e^{2x}$

LHS → $\underbrace{3e^{3x} + 20e^{2x}}_{\frac{dy}{dx}} - 2\underbrace{(e^{3x} + 10e^{2x})}_{y} = (3-2)e^{3x} + \cancel{(20-20)e^{2x}}$

$= 1 \cdot e^{3x}$

$= e^{3x}$

identity: LHS = RHS. ☆. this is how you verify soln to d.e.

Types of solutions:

Trivial Solution: when $y=0$ is a soln for d.e.

Explicit Solution: when soln can be written as $y=f(x)$

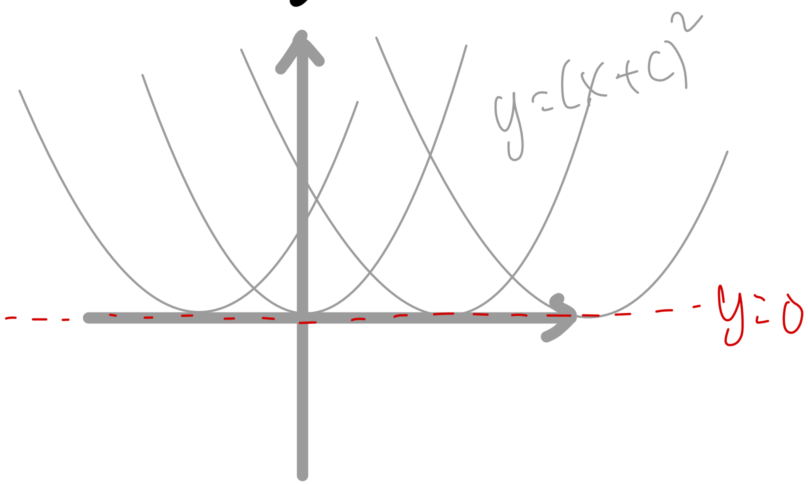
Implicit Solution: when soln can only be written as $G(x,y)=0$

↳ ex: $x^2+xy^2-y=2x$

$y=(x+c)^2$ is a soln for d.e. $(y')^2-4y=0$ for $c\in\mathbb{R}$.

Particular solution: $y=(x+2)^2$ $(c=2)$ or $y=x^2$ $(c=0)$

Singular solution: $y=0$ (singular and trivial solution)



you can not obtain this from $y=(x+c)^2$ and there is only one of them.

$y=c$ is a soln for given d.e. if $c=0$.

General (complete) soln: this applies to linear d.e.

① For example

$y = e^x(\frac{x^2}{2} + c)$ is a general solution for de.

$y' - y - xe^x = 0.$

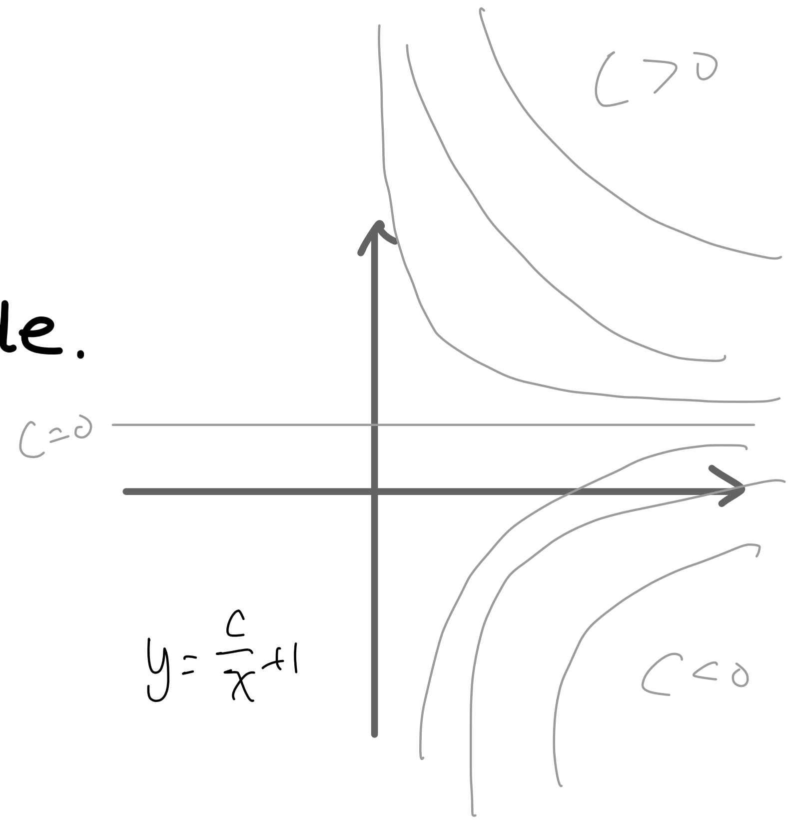
In fact, all solutions look like it.

So it is general and complete.

② For example

$y = \frac{c}{x} + 1$ is a general solution for de.

$xy' + y = 1.$



eg.

(50) Find values of m so that $y=x^{m+1}$ is a solution of the d.e.

$$x^2y''+6xy'+4y=0.$$

$$y=x^{m+1}$$
$$y'=(m+1)x^m$$
$$y''=(m+1)m\,x^{m-1}$$

$$x^a\cdot x^b=x^{a+b}$$

$$x^2\left[m(m+1)x^{m-1}\right]+6x\left[(m+1)x^m\right]+4x^{m+1}=0.$$

$$m(m+1)x^{\underset{a}{2}+\underset{b}{m-1}}+6(m+1)x^{\underset{a}{1}+\underset{b}{m}}+4x^{m+1}=0$$

$$x^{m+1}\qquad x^{m+1}\qquad x^{m+1}$$

$$x^{m+1}\left[m(m+1)+6(m+1)+4\right]=0.$$

$$m(m+1)+6(m+1)+4=0$$
$$m^2+m+6m+6+4=0 \quad \boxed{m^2+7m+10=0}$$

$$am^2+bm+c=0$$
$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$a=1 \quad b=7 \quad c=10$

$$m=\frac{-7\pm\sqrt{49-4\cdot10}}{2}$$
$$=\frac{-7\pm\sqrt{9}}{2}=\frac{-7\pm3}{2}=\begin{cases}\frac{-7+3}{2}=-2\\ \frac{-7-3}{2}=-5\end{cases}$$

$$(m+2)(m+5)=0$$

$$\boxed{m=-2 \text{ or } m=-5}$$

$$y=x^{-2+1}=x^{-1}$$

or

$$y=x^{-5+1}=x^{-4}$$

Initial-Value Problem We are often interested in solving a first-order differential equation*

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

subject to a side condition $y(x_0) = y_0$, where x_0 is a number in an interval I and y_0 is an arbitrary real number. The problem

$$\begin{aligned} &\textit{Solve:} && \frac{dy}{dx} = f(x, y) \\ &\textit{Subject to:} && y(x_0) = y_0 \end{aligned} \tag{2}$$

is called an **initial-value problem** (IVP). The side condition is known as an **initial condition.**

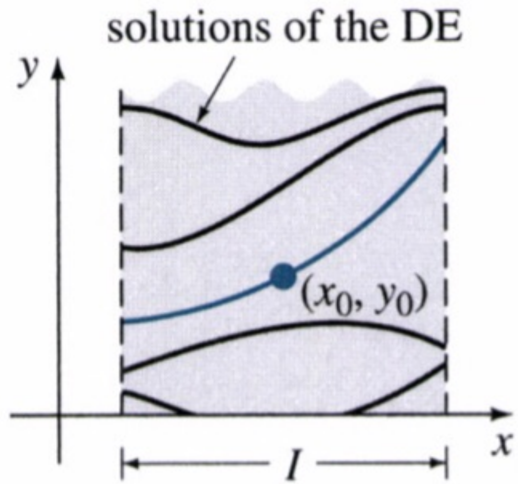


Figure 2.1

THEOREM 2.1 Existence of a Unique Solution

Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b$, $c \leq y \leq d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous on R, then there exist an interval I centered at x_0 and a unique function $y(x)$ defined on I satisfying the initial-value problem (2).

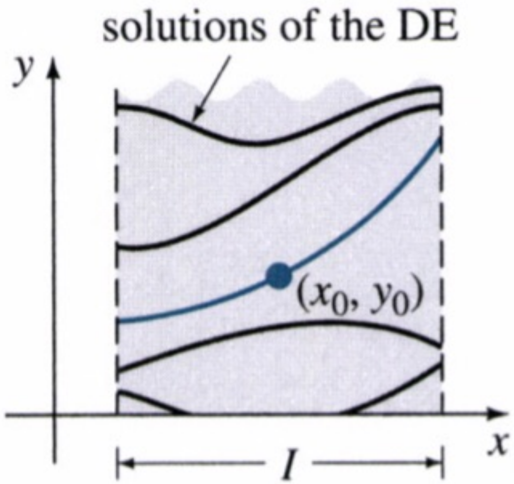


Figure 2.1

HW: Sec 1.1:

1, 3, 5, 11, 13, 15, 21, 35, 49.

2.2 Separable Equation

Methodology of solving 1st-order eq (simplest)

- $\boxed{\text{Solving by Integration}}$

If $g(x)$ is continous

Then 1st-order $\frac{dy}{dx} = g(x)$ canbe solve by integration

$$y = \int g(x)\,dx = G(x) + C$$

$G(x)$ is an antiderivative of $g(x)$.

e.g. $\frac{dy}{dx} = 1 + e^{2x} \longrightarrow y = \int (1+e^{2x})\,dx = x + \frac{1}{2}e^{2x} + C.$

$\frac{dy}{dx} = \sin x \longrightarrow y = \int \sin x\,dx = -\cos x + C.$

DEFINITION 2.1 **Separable Equation**

A differential equation of the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

is said to be **separable** or to have **separable variables.**

observe that

$$h(y)\,\frac{dy}{dx} = g(x).$$

$$h(y)\,dy = g(x)\,dx$$

$$\int h(y)\,dy = \int g(x)\,dx$$

e.g. solve $(1+x)dy - y\,dx = 0$

Dividing by $(1+x)y$,

$\hookrightarrow \dfrac{dy}{y} = \dfrac{dx}{1+x}$

$\int$ $\hookrightarrow \displaystyle\int \frac{dy}{y} = \int \frac{dx}{1+x}$

$$\ln|y| = \ln|1+x| + C.$$

$$y = e^{\ln|1+x| + C}$$

$$= e^{\ln|1+x|} \cdot e^{C}$$

$$= |1+x| \cdot e^{C}$$

$$= \pm(1+x)\,e^{C} = C_1(1+x),$$

solution

$$\boxed{y = C_1(1+x)}$$

e.g. solve. $\frac{dy}{dx} = -\frac{x}{y}$

multiplying $y\,dx$, we can write.

$$y\,dy = -x\,dx$$

$\int$

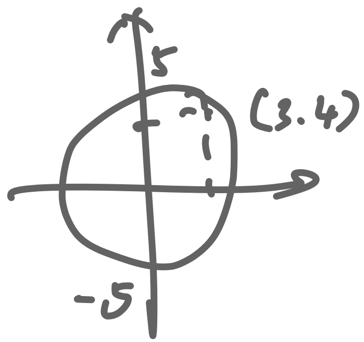
$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$\Leftrightarrow \quad x^2 + y^2 = C^2$$

e.g. solve Initial-Value problem $\frac{dy}{dx} = -\frac{x}{y}$, $y(4)=3$

when $x=4$, $y=3$ $\Rightarrow$ $16+9 = 25 = C^2$.



$\Rightarrow$ This IVP determins $x^2+y^2 = 25$.

☆ losing a solution

The method for separable equations can give us a solution, but it may not give us all the solutions. To illustrate this, consider the eq: $\frac{dy}{dx} = y^{\frac{1}{3}}$.

(a) Use the method of separation of variables to show that

$$y = \left(\frac{2x}{3} + C\right)^{3/2}$$ is a solution.

Indeed, dividing $y^{\frac{1}{3}}$ $\Rightarrow \frac{dy}{y^{\frac{1}{3}}} = dx \Rightarrow \int \frac{dy}{y^{\frac{1}{3}}} = \int dx \Rightarrow \frac{1}{2/3} y^{2/3} = x + C_1$

$$\Rightarrow y = \pm\left(\frac{2}{3}x + \frac{2}{3}C_1\right)^{3/2} = \pm\left(\frac{2x}{3} + c\right)^{3/2}$$

(b) Given IVP $\frac{dy}{dx} = y^{\frac{1}{3}}$ $y(0) = 0$.

show that for $C=0$ the solution $y = \left(\frac{2x}{3}\right)^{3/2}$ for $x \geq 0$ satisfies this IVP. $y = \pm\left(\frac{2x}{3} + c\right)^{3/2} \Rightarrow 0 = y(0) = \left(\frac{2(0)}{3} + c\right)^{3/2} = c^{3/2} \Rightarrow 0 = c$.

$y = (2x/3 + 0)^{3/2} = (2x/3)^{3/2}$ $x \geq 0$

(c) Now show $y \equiv 0$ also satisfies the initial value problem in part (b). Hence, IVP does not have a unique solution.

$$y(x) \equiv 0$$

$$\frac{dy}{dx} = y^{\frac{1}{3}} = 0^{\frac{1}{3}} = 0$$

$$y(0) = (0)^{\frac{1}{3}} = 0$$

(d) Finall, show that the condition of the (existance and uniquness) theorem are not satisfied.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y^{\frac{1}{3}}\right) = \frac{1}{3}y^{-2/3} = \frac{1}{3y^{2/3}}$$

since $\frac{\partial f}{\partial y}$ is not continuous when $y=0$, there is no rectangle, containing the point (0.0), in which both f and $\frac{\partial f}{\partial y}$ are continuous. Therefore, the Thm does not apply to this IVP.

12 $\dfrac{dx}{dy} = \dfrac{1+2y^2}{y\sin x} = \dfrac{\frac{1+2y^2}{y}}{\sin x}.$ separable

multiply

$\sin x\, dy.$

$$\sin x\, \cancel{dy}\, \frac{dx}{\cancel{dy}} = \frac{1+2y^2}{y\, \cancel{\sin x}}\, \cancel{\sin x}\, dy.$$

$$\sin x\, dx = \frac{1+2y^2}{y}\, dy.$$

Integration

$$\underline{\int \sin x\, dx} = \int \frac{1+2y^2}{y}\, dy = \int \frac{1}{y} + 2y\, dy.$$

$$\boxed{-\cos x = \ln|y| + y^2 + c.}$$ solution

$$x = \arccos\left(-\left[\ln|y| + y^2 + c\right]\right)$$

$$\boxed{\frac{dx}{dy} = \frac{x^2 y}{1+x}} \longrightarrow -\frac{1}{x} + \ln|x| = \frac{1}{2}y^2 + c$$

$$\frac{dx}{dy} = \boxed{\frac{x^2 y}{1+x}.} = \frac{y}{\frac{1+x}{x^2}}.$$

multiply. $\frac{1+x}{x^2}\, dy$ $\quad * \; x \neq 0.$

$$\frac{1+x}{x^2} \cancel{dy} \cdot \frac{dx}{\cancel{dy}} = \frac{1+\cancel{x}}{\cancel{x^2}} dy \cdot \frac{\cancel{x^2} y}{1+\cancel{x}}$$

$$\frac{1+x}{x^2} dx = y\, dy.$$

integration

$$\int \frac{1+x}{x^2}\, dx = \int y\, dy.$$

$$\int \frac{1}{x^2} + \frac{1}{x}\, dx = \int y\, dy$$

$$\boxed{-\frac{1}{x} + \ln|x| = \frac{1}{2}y^2 + C}$$

Solution is founded by the method for separation of variables.

R

$\boxed{x = 0.}$ $\quad dx = 0$

$$\frac{dx}{dy} = 0.$$

$$\frac{x^2 y}{1+x} = \frac{0 \cdot y}{1+0} = 0.$$

Homework

2.2.: 1, 3, 5, 7, 9, 11, 13, 15, 41, 43, 45

2.3 Homogeneous Eq :

DEFINITION 2.2 Homogeneous Function

If a function f has the property that

$$f(tx, ty) = t^n f(x, y) \quad (1)$$

for some real number n, then f is said to be a **homogeneous function of degree n.**

$f(x,y) = x^2 - 3xy + 5y^2$

$f(tx, ty) = (tx)^2 - 3(tx)(ty) + 5(ty)^2 = t^2(x^2 - 3xy + 5y^2)$.

$= t^2 f(x.y)$.

$f(tx. ty) = t^2 f(x.y)$.

$f(x.y)$ homogeneous functn. of degree 2

$g(tx, ty) = g(x, y)$.

$= t^0 g(x.y)$

$= 1 \cdot g(x.y)$

$= g(x, y)$.

$g(x,y)$ homogeneous functions of degree 0.

$$f(x,y) = x^2 + xy + 2x + 1.$$

$$f(tx, ty) = (tx)^2 + (tx)(ty) + 2(tx) + 1.$$

$$= t^2x^2 + t^2xy + 2tx + 1.$$

$$\neq t^2 f(x,y) = t^2(x^2 + xy + 2x + 1)$$

$$= t^2x^2 + t^2xy + 2t^2x + t^2.$$

$$\neq t f(x,y) = tx^2 + txy + 2tx + t.$$

$f(x,y) \neq t^n f(x,y)$ for any real number n.

$f(x,y)$ is **NOT** a homogeneous function.

DEFINITION 2.3 Homogeneous Equation

A differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad (3)$$

is said to be **homogeneous** if both coefficients M and N are homogeneous functions of the same degree.

$M(x, y) \rightarrow$ homogeneous same degree

$N(x, y) \rightarrow$

$$\underbrace{(x+y)}\,dx + \underbrace{y}\,dy = 0.$$

$M(tx, ty) = t M(x, y)$. $\quad N(tx, ty) = t N(x, y)$.

M and N degree 1. homogeneous eq.

$$(x+y)\,dx + \underbrace{xy}\,dy = 0$$

$M(tx, ty) = t M(x, y)$. $\quad N(tx, ty) = t^2 N(x, y)$

degree $= 1$ $\qquad$ degree $= 2$.

Solving a homogeneous DE.

Substituting either $\boxed{y = ux}$ or $\boxed{x = vy}$

where u and v are new dependent variables,

will reduce the eq to a separable first order DE

Q: Which one should we choose? $y = ux$ or $x = vx$?

An: $Mdx + Ndy = 0$

M is simpler $\Rightarrow$ $x = vy$

N is simpler $\Rightarrow$ $y = ux$

Example : $(x-y)dx + \underbrace{x}\,dy = 0.$

$\underline{N(x,y)}$ looks simpler. Use $\boxed{y=ux}$.

$dy = x\,du + u\,dx$

(use produce rule).

$$(x-ux)\,dx + x(x\,du+u\,dx) = 0.$$

$$x\,dx \underline{-ux\,dx} + x^2 du + \underline{xu\,dx} = 0$$

$$x\,dx + x^2 du = 0.$$

$$x\,dx = \underline{-x^2 du}$$

$$-\frac{x}{x^2}\,dx = du.$$

$$\int -\frac{1}{x}\,dx = \int du.$$

$$-\ln|x| = u + C.$$

$$u = \frac{y}{x}.$$

$$-\ln|x| = \frac{y}{x} + C.$$

$$-x\ln|x| = y + Cx$$

$$y = -x\ln|x| - Cx.$$

$\square$

$(x-y)\,dx + x\,dy = 0$

if we replace $\boxed{x = vy}$: $dx = v\,dy + y\,dv.$ $v = \frac{x}{y}$

$(vy - y)\,dx + x\,dy = 0$

$(vy - y)(v\,dy + y\,dv) + (vy)\,dy = 0.$

$(v^2y - \cancel{yv} + \cancel{vy})\,dy + (vy^2 - y^2)\,dv = 0.$

$v^2y\,dy + (vy^2 - y^2)\,dv = 0.$

$v^2y\,dy = (y^2 - vy^2)\,dv$

$\frac{y}{y^2}\,dy = \frac{1-v}{v^2}\,dv.$

$\int \frac{1}{y}\,dy = \int \frac{1-v}{v^2}\,dv.$

$\ln|y| = \int \frac{1}{v^2} - \frac{1}{v}\,dv$

$= -\frac{1}{v} - \ln|v| + C.$

$\ln|y| = -\frac{y}{x} - \ln\left|\frac{x}{y}\right| + C.$

$\frac{y}{x} = -\ln|y| - \ln\left|\frac{x}{y}\right|$

$= -\ln\left|y \cdot \frac{x}{y}\right| + C$

$= -\ln|x| + C$

$\frac{y}{x} = -\ln|x| + C$

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = -\frac{\cancel{x^n} M(1,\frac{y}{x})}{\cancel{x^n} N(1,\frac{y}{x})} = \frac{-M(1,\frac{y}{x})}{N(1,\frac{y}{x})}.$$

$$x\frac{dy}{dx} = y + \cancel{x}e^{y/x}, \quad y(1) = 1.$$

$y = ux$

$$\boxed{u = \frac{y}{x}}$$

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

$$= u + e^{u}.$$

$dy = u\,dx + x\,du$

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

$$\cancel{u} + x\frac{du}{dx} = \cancel{u} + e^{u}$$

$$x\frac{du}{dx} = e^{u}.$$

$$\int e^{-u}\,du = \int \frac{1}{x}\,dx$$

$$-e^{-u} = \ln|x| + c.$$

$$-e^{-y/x} = \ln|x| + C.$$

$x = 1 \quad y = 1$

$$-e^{-1} = 0 + C$$

$$C = -e^{-1}$$

$$\boxed{e^{-1} - e^{-y/x} = \ln|x|}$$

$\boxed{2.4}$ $$\frac{dy}{dx} = \frac{y}{x} + \frac{\cancel{x}^2}{y^2} + 1.$$

$$\frac{y}{x} = u. \quad y = ux$$

$$dy = u\,dx + x\,du.$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\cancel{u} + x\frac{du}{dx} = \cancel{u} + \frac{1}{u^2} + 1.$$

$$x\frac{du}{dx} = \frac{1+u^2}{u^2}.$$

$$\frac{u^2}{1+u^2}du = \frac{1}{x}dx.$$

$$\int \frac{u^2+1}{1+u^2} - \frac{1}{1+u^2}du = \int \frac{1}{x}dx$$

$$\int 1 - \frac{1}{1+u^2}du = \ln|x| + C.$$

$$u - \arctan u = \ln|x| + C$$

$$\frac{y}{x} - \arctan\frac{y}{x} = \ln|x| + C.$$

Solve $x\frac{dy}{dx} = y + xe^{y/x}$. $y(1) = 1$

$$xdy - (y + xe^{y/x})dx = 0$$

$$M(tx, ty) = ty + tx\, e^{ty/tx} = t(y + xe^{y/x}) = tM(x,y)$$

$$N(tx, ty) = tx = tN(x,y)$$ both degree 1.

$y = ux$. $u = \frac{y}{x}$ $dy = xdu + udx \Rightarrow \frac{dy}{dx} = x\frac{du}{dx} + u$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}e^{y/x}$$

$$x\frac{du}{dx} + u = u + \frac{1}{u}e^{u}$$

$$x\frac{du}{dx} = \frac{1}{u}e^{u}$$

$$e^{-u}du = \frac{dx}{x}$$

$$\int e^{-u}du = \int \frac{1}{x}dx$$

$$-e^{-u} + C = \ln|x|$$

$$-e^{-\frac{y}{x}} + c = \ln|x|$$

$y = 1$, $x = 1$.

$$\Rightarrow -e^{-1} + c = \ln 1 = 0$$

$$c = e^{-1}$$

$$\boxed{-e^{-\frac{y}{x}} + e^{-1} = \ln|x|}$$

$$\underbrace{(x^2+y^2)}_{M}\,dx + \underbrace{3xy}_{N}\,dy = 0$$

(32) $\boxed{(x^2+2y^2)dx = xy\,dy}$

$M(tx,ty) = t^2x^2+t^2y^2$

$= t^2(x^2+y^2)$

$= t^2 M(x,y)$

$N(tx,ty) = 3tx\,ty$

$= 3t^2xy$

$= t^2\,3xy$

$= t^2N(x,y)$

degree 2.

homogeneous.

① substitution $\boxed{y=ux}$ $dy = u\,dx + x\,du$.

$$(x^2+u^2x^2)\,dx + 3x(ux)(u\,dx + x\,du) = 0$$

$$\underline{x^2dx + u^2x^2dx + 3u^2x^2dx} + 3x^3u\,du = 0$$

② separation.

$$\frac{(x^2+u^2x^2+3u^2x^2)dx}{x^2} + \frac{3x^3u\,du}{x^2} = 0.$$

$$(1+u^2+3u^2)dx + 3ux\,du = 0.$$

$$\frac{(1+4u^2)}{1+4u^2}dx + \frac{3xu\,du}{1+4u^2} = 0$$

$$dx + 3\frac{u}{1+4u^2}x\,du = 0$$

dividing x

$$\frac{1}{x}dx + \frac{3u}{1+4u^2}du = 0 \qquad \int -\frac{1}{x}dx = \int \frac{3u}{1+4u^2}du$$

$$\int -\frac{1}{x}dx = \int \frac{3u}{1+4u^2}du.$$

$w = 1+4u^2$

$dw = 8u\,du.$

$\frac{1}{8}dw = u\,du$

$\frac{3}{8}dw = 3u\,du.$

$$-\ln|x|+c = \int \frac{\frac{3}{8}dw}{w}$$

$$-\ln|x|+c = \frac{3}{8}\int \frac{1}{w}dw$$

$$= \frac{3}{8}\ln|w|$$

$$= \frac{3}{8}\ln|1+4u^2|.$$

$u = \frac{y}{x}$ $\quad y = ux$

$$\boxed{-\ln|x|+c = \frac{3}{8}\ln\left|1+4\frac{y^2}{x^2}\right|}$$ ← solution

$$|x|^{-1}\cdot e^{c} = e^{\ln\left|1+4\frac{y^2}{x^2}\right|^{\frac{3}{8}}} = \left|1+4\frac{y^2}{x^2}\right|^{\frac{3}{8}}$$

$$|x|^{-\frac{8}{3}}\cdot C_1 = \left|1+4\frac{y^2}{x^2}\right| = \left(1+4\frac{y^2}{x^2}\right)$$

$\times x^2$ $\quad |x|^{-\frac{8}{3}}|x|^2 C_1 = x^2+4y^2$

$$|x|^{-\frac{8}{3}+\frac{6}{3}}C_1 = x^2+4y^2$$

$$C_1|x|^{-\frac{2}{3}} = x^2+4y^2.$$

$$\boxed{C_1(x^2)^{-\frac{2}{3}} = x^2+4y^2}$$

例2 $(x^2+2y^2)dx = xy\,dy$

25 Linear Equation

2.5 Linear Equations 1st.

$$\underline{a_1(x)}\frac{dy}{dx} + a_0(x)y = g(x).$$

$$\frac{dy}{dx} + P(x)y = f(x).$$

idea: convert a nonexact eq $\rightarrow$ an exact eq.

DEFINITION 2.5 Linear Equation

A differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is said to be a **linear equation.**

Dividing by the lead coefficient $a_1(x)$ gives a more useful form, the **standard form,** of a linear equation:

$$\frac{dy}{dx} + \underline{P(x)}y = f(x). \tag{1}$$

making sure the form is correct before choosing P(x)

② $\boxed{\text{Integrating Factor}}$

$$\frac{dy}{dx} + p(x)y = f(x).$$

$$dy + p(x)y\,dx = f(x)\,dx.$$

$$dy + [p(x)y - f(x)]dx = 0 \cdots\cdots ①$$

a function $\mu(x)$ make ① exact

i.e. $$\underbrace{\mu(x)}_{N}dy + \underbrace{\mu(x)[p(x)y - f(x)]}_{M}dx = 0 \cdots\cdots ②.$$

② is exact $\Longleftrightarrow$ $$\frac{\partial M}{\partial y} = \frac{\partial \mu(x)[p(x)y - f(x)]}{\partial y} = \frac{\partial \mu(x)}{\partial x} = \frac{\partial N}{\partial x}$$

$$\mu(x)p(x) = \frac{\partial \mu(x)}{\partial x} = \frac{d\mu}{dx}$$

$$\int p(x)\,dx = \int \frac{1}{\mu}\,d\mu. \qquad \text{separable.}$$

$$\int p(x)\,dx = \ln|\mu|.$$

$$|\mu| = e^{\int p(x)dx}$$

$$\Rightarrow \boxed{\mu(x) = e^{\int p(x)dx}} \leftarrow \text{Integrating factor.}$$

$$\frac{dy}{dx} + P(x)y = f(x)$$

SOLVING A LINEAR FIRST-ORDER EQUATION

(*i*) To solve a linear first-order equation first put it into the form (1); that is, make the coefficient of dy/dx unity.

(*ii*) Identify $P(x)$ and find the integrating factor

$$e^{\int P(x)\,dx}.$$

(*iii*) Multiply the equation obtained in step (*i*) by the integrating factor:

$$e^{\int P(x)\,dx}\frac{dy}{dx} + P(x)e^{\int P(x)\,dx}y = e^{\int P(x)\,dx}f(x).$$

(*iv*) The left side of the equation in step (*iii*) is the derivative of the product of the integrating factor and the dependent variable y; that is,

$$\boxed{\frac{d}{dx}\left[e^{\int P(x)\,dx}y\right] = e^{\int P(x)\,dx}f(x).}$$

(*v*) Integrate both sides of the equation found in step (*iv*).

from step v) you get

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}f(x)\,dx$$

$$\boxed{y = \frac{\int e^{\int P(x)dx}f(x)\,dx}{e^{\int P(x)dx}}}$$

Summary

① put $\frac{dy}{dx} + P(x)y = f(x)$ (standard form).

coefficient 1.

② $P(x) = ?$ $\mu(x) = e^{\int P(x)dx} = ?$ (integrating factor)

③ $y(x) = \frac{1}{\mu(x)} \left[\int \mu(x) f(x) dx + C \right]$

or $\frac{d}{dx} \mu(x) y = \mu(x) f(x)$

Solve $\frac{dy}{dx} - 3y = 0$ $\qquad$ $\frac{dy}{dx} + P(x)\,y = f(x)$ standard.

Step ① $P(x) = -3.$ $\qquad$ $f(x) = 0.$

Step ② $\mu(x) = e^{\int P(x)dx} = e^{\int -3\,dx} = e^{-3x \boxed{\cancel{+C}}}$

$= e^{-3x}$

← We can drop the constant because it'll only give as another constant in the end (C in the answer)

Step ③ $\boxed{\frac{d}{dx}\mu(x)\,y = \mu(x)f(x) = 0}$

$$\int d\,\mu(x)\,y = \int 0\,dx$$

$$\mu(x)\,y = C$$

$$e^{-3x}y = C$$

$$y = \frac{C}{e^{-3x}} = Ce^{3x}$$

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)f(x)dx + C\right]$$

$$= \frac{1}{e^{-3x}}\left[\underbrace{\int e^{-3x}\cdot 0\,dx}_{=0} + C\right]$$

$$= \frac{C}{e^{-3x}} = Ce^{3x}$$

Example $y' + (\tan x) y = \cos^2 x,\ y(0) = -1$

* make sure you check if the given d e is linear. It is a good habit

equation is already in the standard form $P(x) = \tan x$

Integrating Factor $e^{\int P(x)dx} = e^{\int \tan x\, dx}$

$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$ let $u = \cos x$ $du = -\sin x\, dx$

$= \int -\frac{1}{u} du$

$= -\ln|u|$

$= -\ln|\cos x|$ ← We can drop the constant c because it'll only give us another constant at the end (in the answer)

$e^{\int \tan x dx} = e^{-\ln|\cos x|} = e^{\ln|1/\cos x|} = \frac{1}{\cos x}$

$\frac{d}{dx}\left(\frac{1}{\cos x} y\right) = \frac{1}{\cos x}\cos^2 x$

$\int d \frac{1}{\cos x} y = \int \cos x\, dx$

$\frac{1}{\cos x} y = \sin x + c$

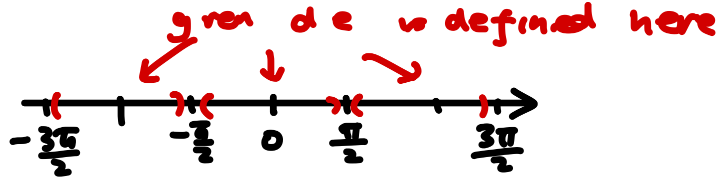
$y = \cos x(\sin x + c)$

$= \cos x \sin x + c\cos x$

Since $y(0) = -1$, $-1 = \cos 0 \sin 0 + c\cos 0 = c$, $c = -1$

so, $y = \sin x \cos x - \cos x$ on the interval $\underline{-\frac{\pi}{2} < x < \frac{\pi}{2}}$ ← Why?

note that $y' + \underbrace{\tan x\, y}_{\tan x \text{ is defind whenever } \cos x \neq 0} = \cos^2 x$



since the ivp we are given includes $x=0$, we choose interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ so whenever $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the solution we have is the solution to d e

Example (simple version) $x(x-2)y' + 2y = 0$ this is not in standard form

$y' + \frac{2}{x(x-2)}y = 0$ (standard form) and IF $\underbrace{e^{\int \frac{2}{x(x-2)}dx}}_{\text{use fraction decomposition}}$

$\frac{2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2} \rightarrow 2 = A(x-2) + Bx$ when $x=2$ $2=2B$, $B=1$

$x=0$ $2=-2A$, $A=-1$

$= -\frac{1}{x} + \frac{1}{x-2}$

So, IF is $e^{\int \frac{2}{x(x-2)}dx} = e^{\int -\frac{1}{x} + \frac{1}{x-2}dx} = e^{-\ln|x| + \ln|x-2|} = e^{\ln|\frac{1}{x}|}e^{\ln|x-2|}$

$= \frac{1}{|x|}|x-2| = \left|\frac{x-2}{x}\right|$

It will always simplify to

$\left[\frac{d}{dx}(IF\ y) = IF\ f(x)\right]$

so, $x(x-2)y' + 2y = 0$ become $\frac{d}{dx}\left(\left|\frac{x-2}{x}\right| y\right) = 0$

$\int d\left|\frac{x-2}{x}\right| y = \int 0\,dx$

$\left|\frac{x-2}{x}\right| y = c$

$y = c\left|\frac{x}{x-2}\right| = \tilde{c}\frac{x}{x-2}$

2.4. Exact Equations

DEFINITION 2.4 **Exact Equation**

A differential expression

$$M(x, y)\,dx + N(x, y)\,dy$$

is an **exact differential** in a region R of the xy-plane if it corresponds to the total differential of some function $f(x, y)$. A differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation** if the expression on the left side is an exact differential.

total differential

shows up in Calc4

if $z=f(x,y)$ then total diff. is

$$dz=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

partial derivative of f with respect to x and y

THEOREM 2.2 **Criterion for an Exact Differential**

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b, c < y < d$. Then a necessary and sufficient condition that

$$M(x, y)\,dx + N(x, y)\,dy$$

* Make sure d.e is of this form.

be an exact differential is

mindful of the sign!

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{4}$$

We need to find $f(x,y)$ such that

$$\frac{\partial f}{\partial x}=M(x,y),\ \frac{\partial f}{\partial y}=N(x,y)$$

example: $\overset{M(x,y)}{(5x+4y)}\,dx + \overset{N(x,y)}{(4x-8y^3)}\,dy = 0$

Exact since $M_y = 4 = N_x = 4$.

We need to find $f(x,y)$ such that $f_x = M$, $f_y = N$

(this step is different from checking if given d.e. is exact).

① If $f_x = M$, then $f = \int f_x\,dx = \int M\,dx$

so $f = \int \underbrace{5x+4y\,dx}_{\text{integrate with respect to } x \text{ so that } y \text{ like a constant}} = \frac{5x^2}{2} + 4yx + \underbrace{C(y)}_{\text{any term which only depends on } y \text{ will be "constant term" since } f \text{ is a function with TWO variables.}}$

② $f = \frac{5x^2}{2} + 4yx + c(y)$ and $f_y = N$

$f_y = 0 + 4x + c'(y) = 4x - 8y^3 = N$

so $c'(y) = -8y^3$ and we need to find $c(y)$.

$c(y) = \int c'(y)\,dy = \int -8y^3\,dy = -2y^4 + C$

this is constant.

③ solution $\frac{5x^2}{2} + 4yx - 2y^4 + C = 0$

this constant can be computed if we have IVP (initial value problem)

Example: $(2y^2x-3)dx = (2yx^2+4)\,dy$

What ~~Not~~ to do: $\underbrace{(2y^2x-3)}_{M}dx = \underbrace{(2yx^2+4)}_{N}dy$

$M_y = 4yx = N_x = 4yx \longrightarrow$ exact d.e.!

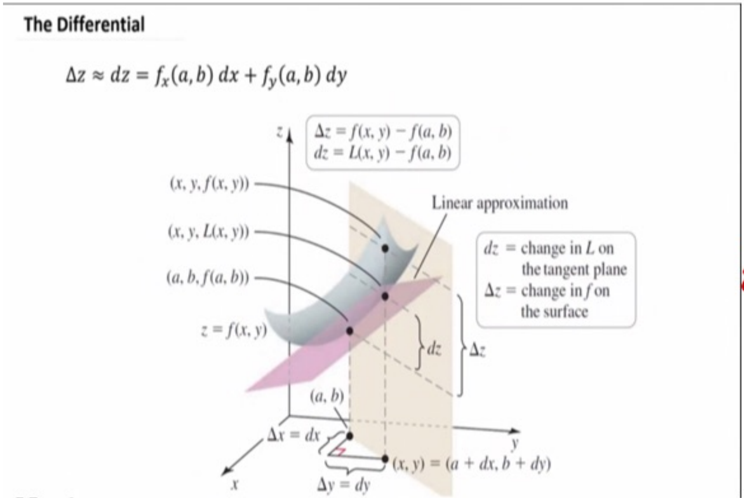
What to do: $(2y^2x-3)dx-(2yx^2+4)dy=0$

$$\underbrace{(2y^2x-3)}_{M}dx + \underbrace{(-2yx^2-4)}_{N}dy=0$$

$M_y = 4xy \neq N_x = -4xy \longrightarrow$ Not Exact d.e.

"Cal" $\quad z = f(x,y)$

$$dz = df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x dx + f_y dy .$$



$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + \underline{f(a,b)}$$

$$\underbrace{z - f(a,b)}_{\Delta z} = f_x(a,b)\underbrace{(x-a)}_{\Delta x} + f_y(a,b)\underbrace{(y-b)}_{\Delta y} .$$

$$dz = f_x(a,b)\,dx + f_y(a,b)\,dy \quad 20 .$$

$$df = 0 .$$

$\Leftrightarrow f =$ constant.

$\boxed{\#3}$

2.6 Bernoulli Equation

Bernoulli's Equation The differential equation

this is not a linear eq

$$\frac{dy}{dx} + P(x)y = f(x)y^n, \tag{1}$$

where n is any real number, is called **Bernoulli's equation.** For $n = 0$ and $n = 1$, equation (1) is linear in y. Now for $y \neq 0$, (1) can be written as

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = f(x). \tag{2}$$

If we let $w = y^{1-n}$, $n \neq 0$, $n \neq 1$, then

$$\frac{dw}{dx} = (1 - n)y^{-n}\frac{dy}{dx}.$$

With these substitutions, (2) can be simplified to the linear equation

this is a linear de

$$\frac{dw}{dx} + (1 - n)P(x)w = (1 - n)f(x). \tag{3}$$

Solving (3) for w and using $\underline{y^{1-n} = w}$ leads to a solution of (1).

↳ the answer will contain w you 'must' change w to y^{1-n} at the end

Use the method discussed under "Bernoulli Equations" to solve the equation.

$$\frac{dy}{dx}+\frac{dy}{x-2}=5(x-2)y^{\frac{1}{2}}$$

Divide by $y^{\frac{1}{2}}$ *

$$y^{-\frac{1}{2}}\frac{dy}{dx}+\frac{1}{x-2}y^{\frac{1}{2}}=5(x-2)$$

$$v=y^{\frac{1}{2}}$$

$$v^2=y$$

$$2v\frac{dv}{dx}=\frac{dy}{dx}$$

$$2\frac{dv}{dx}+\frac{1}{x-2}v=5(x-2)$$

$$\frac{dv}{dx}+\frac{1}{2(x-2)}v=\frac{5(x-2)}{2}$$

$$\mu(x)=e^{\int\frac{1}{2(x-2)}dx}=\sqrt{|x-2|}$$

$$v(x)=\frac{1}{\sqrt{|x-2|}}\int\frac{5(x-2)\sqrt{|x-2|}}{2}dx=\frac{1}{\sqrt{|x-2|}}\left(|x-2|^{5/2}+c\right)$$

$$=\frac{1}{\sqrt{|x-2|}}\int\frac{5}{2}|x-2|^{\frac{3}{2}}dx$$

$$=\frac{1}{\sqrt{|x-2|}}\int\frac{5}{2}\cdot\frac{2}{5}d|x-2|^{\frac{5}{2}}$$

$$=(x-2)^2+c|x-2|^{-1/2}$$

Answer: $y=v^2=\left[(x-2)^2+c|x-2|^{-1/2}\right]^2$ * and $y\equiv 0$.

don't memorize the equation (3) but rather, the process of solving Bernoulli de